1. Introduction. Brownian Motion.

Dienstag, 9. Oktober 2018 13:23

Quantum optics.
Leading textbooks include

- Scully/Zubairy: Quantum Optics
- Loudon: Quantum Theory of Light
- Louisell: Quantum statistical properties of Radiation
- Walls/Milburn: Quantum Optics
- Agarwal: Quantum Optics
- Schleich: Quantum Optics in Phase Space

Quantum dissipation
Leading textbooks include

- Breuer/Petruccione: Open quantum systems
- Weiss: Quantum dissipation
- Gardiner/Zoller: Quantum noise
- Carmichael: Statistical methods in quantum optics
- Barnett/Radmore: Methods in Theoretical quantum optics

- non-classical light emission (single photons, superbunching)
- non-equilibrium phonon dynamics
- polarization-entangled photon pairs
- electrically-pumped single photon sources
- indistinguishable photons
- phonon lasing (acoustic cavities)

- [illegible]
- phonon lasing (acoustic cavities)
- optomechanical systems (solid-state based, CQED)

1.) Brownian motion (Langevin / quantum Langevin)
2.) Quantum master equation (Lindblad)
3.) Quantum Stochastic Schrödinger Equation. Quantum Monte-Carlo simulation.
4.) Propagator method. Feynman-Vernon influence functional.
5.) Independent boson model.
6.) Caldeira-Leggett master equation.
7.) Quantum optical master equation. Mollow physics.
8.) Quantum feedback. single-photon limit.
9.) Quantum feedback. Matrix-product state.

Brownian motion:

$$m\ddot{x} = m\dot{v} = -\gamma m v + R(t)$$

equation of motion for a Brownian particle subjected to a random force

$R(t)$ is assumed to be not known, but we can find properties in the average sense.

(i) $\langle R(t) \rangle = 0$, (ii) $\langle R(t)\, v(t) \rangle = 0$, (iii) $\langle R(t) R(t') \rangle = 2\pi R_0\, \delta(t-t')$

what does the random force for the temporal average of the variance?

$$\lim_{t \to \infty} \left[\frac{1}{t} \langle [x(t) - x(0)]^2 \rangle \right] = ?$$ this will yield the Green-Kubo formula.

[illegible] with the identity $\boxed{x(t) - x(0) = \int^{t} dt'\, \dot{x}(t')}$ $[v(t) = \dot{x}(t)]$

$t \to \infty$ — — Green–Kubo formula.

start with the identity $\boxed{x(t) - x(0) = \int_0^t dt' \, \underline{\underline{\dot{x}(t')}}}$ $\quad [\dot{x}(t) = \dot{x}(t)]$

$$[x(t)-x(0)]^2 = \int_0^t dt' \int_0^t dt'' \; \dot{x}(t') \, \dot{x}(t'')$$

$$= \int_0^t dt' \int_0^{t'} dt'' \; \dot{x}(t') \, \dot{x}(t'') + \int_0^t dt'' \int_0^{t''} dt' \; \dot{x}(t') \dot{x}(t'')$$

$$= \{ \text{relabelling} \} = 2 \int_0^t dt' \int_0^{t'} dt'' \; \dot{x}(t') \dot{x}(t'')$$

$$= \{ t'' = t' - s, \; s = t' - t'', \; -ds = dt'' \}$$

$$= 2 \int_0^t dt' \int_{t'}^{0} (-ds) \; \dot{x}(t') \, \dot{x}(t'-s)$$

now we use the observation that we have a stationary on average.

$$\langle \dot{x}(t') \, \dot{x}(t'-s) \rangle = \langle \dot{x}(0) \, \dot{x}(-s) \rangle = \langle \dot{x}(s) \, \dot{x}(0) \rangle$$

$$\langle [x(t) - x(0)]^2 \rangle = 2 \int_0^t dt' \int_0^{t'} ds \; \langle \dot{x}(s) \, \dot{x}(0) \rangle$$

use the following integration in parts

$$\int_0^t dt' \; F(t') = \int_0^t dt' \left(\frac{d}{dt'} t' \right) F(t') = \underbrace{t' F(t') \Big|_0^t}_{= t F(t)} - \int_0^t dt' \, t' \left(\frac{d}{dt'} F(t') \right)$$

$$\frac{1}{t} \int_0^t dt' \; F(t') = F(t) - \int_0^t dt' \; \frac{t'}{t} \, \frac{dF(t')}{dt'}$$

$$F(t') = \int_0^{t'} ds \, \langle \dot{x}(s) \dot{x}(0) \rangle \qquad \frac{d}{dt'} F(t') = \langle \dot{x}(t') \dot{x}(0) \rangle$$

$$\Rightarrow \; \frac{1}{t} \int_0^{t} dt' \int_0^{t'} ds \, \langle \dot{x}(s) \dot{x}(0) \rangle = \int_0^t ds \, \langle \dot{x}(s) \dot{x}(0) \rangle - \int_0^t dt' \, \frac{t'}{t} \, \langle \dot{x}(t') \dot{x}(0) \rangle$$

$$= \int_0^t dt' \left[1 - \frac{t'}{t} \right] \langle \dot{x}(t') \dot{x}(0) \rangle$$

take the limit $t \to \infty$:

$$\lim \left[\frac{1}{t} \langle [x(t) - x(0)]^2 \rangle \right] = \lim \int^t dt' \left[1 - \frac{t'}{t} \right] \langle \dot{x}(t') \dot{x}(0) \rangle$$

$$\underbrace{\lim_{t\to\infty}\left[\frac{1}{t}\langle[x(t)-x(0)]^2\rangle\right]}_{\text{our observation}} = \lim_{t\to\infty}\int_0^t dt'\left[1-\frac{t'}{t}\right]\langle\dot{x}(t')\dot{x}(0)\rangle$$

$$= \int_0^\infty dt' \langle\dot{x}(t')\dot{x}(0)\rangle$$

the velocity autocorrelation is a measure for the mean variance in space if a random force acts on the particle. therefore, the particle is not just damped but also transported in space : damping leads to motion

Fluctuation-Dissipation Theorem

Green-Kubo-formel : $\lim_{t\to\infty}\left[\frac{1}{t}\langle\Delta A(t)\rangle = \int_0^\infty ds\, \langle\dot{A}(s)\dot{A}(0)\rangle\right.$

<u>note</u> : · the fluctuation-dissipation theorem is the closest thing to a fundamental law in non-equilibrium phenomena.

· the stochastic process reproduces at least a thermal equilibrium on average

$$m\dot{v} = -\gamma m v + R(t)$$

$$v(t) = e^{-\gamma t}v(0) + \int_0^t dt'\, e^{-\gamma(t-t')}\frac{R(t')}{m}$$

$$[v(t)]^2 = e^{-2\gamma t}\left[v_0^2 + 2\int_0^t dt\, e^{\gamma t'}\frac{v(0)R(t')}{m} + \int_0^t dt'\int_0^t dt''\, e^{\gamma(t'+t'')}\frac{R(t')R(t'')}{m^2}\right.$$

take the average over all possible initial states.

$$\langle v^2(t)\rangle = e^{-2\gamma t}\left[\langle v_0^2\rangle + \frac{2\pi R_0}{m^2}\frac{e^{2\gamma t}-1}{2\gamma}\right]$$

$$\xrightarrow{t\to\infty} \frac{\pi R_0}{m^2\gamma}, \quad \lim_{t\to\infty}\langle v^2(t)\rangle = \frac{2}{m}\frac{3}{2}k_B T = \frac{3k_BT}{m}$$

$$\left(U_{KIN} = U_{THERM} = \tfrac{1}{2}m\langle v^2\rangle = \tfrac{3}{2}k_BT\right)$$

$$\frac{\pi R_0}{m^2\gamma} = \frac{3k_BT}{m} \Rightarrow \gamma = \frac{R_0\pi}{3mk_BT}$$

$$\boxed{\langle R(t')R(t'')\rangle = 6m k_B T \gamma\, \delta(t'-t'')}$$

(ill-defined mathematically)

given these properties, the spatial mean variance can be calculated.

$\langle (x(t)-x(0))^2\rangle = \langle x^2(t)\rangle$, we make the particle always start at $x(0)=0$

we need an equation of motion for $x^2(t)$:

$$x\dot{x} = \frac{1}{2}\frac{d}{dt}x^2 \quad , \quad x\ddot{x} + \underbrace{\dot{x}^2 - \dot{x}^2}_{=0} = \frac{d}{dt}(x\dot{x}) - \dot{x}^2$$

$$= \frac{d}{dt}\left(\frac{1}{2}\frac{d}{dt}x^2\right) - \dot{x}^2$$

$$\frac{d^2}{dt^2}(x^2) = 2x\ddot{x} + 2\dot{x}^2 = 2x\ddot{x} + 2v^2(t)$$

so $\frac{d}{dt^2}\langle x^2\rangle = 2\langle x\left[-\gamma v + \frac{R(t)}{m}\right]\rangle + 2\langle e^{-2\gamma t}v(0) + \frac{\pi R_0}{m^2\gamma}(1-e^{-2\gamma t})\rangle$

$$= -2\gamma \underbrace{\langle x(t)\, v(t)\rangle}_{=\frac{1}{2}\frac{d}{dt}\langle x^2\rangle} + \frac{2}{m}\underbrace{\langle xR(t)\rangle}_{=0} + 2\underline{\underline{e}}^{-2\gamma t}\langle v(0) + \frac{2\pi R_0}{m^2\gamma}(1-\underline{\underline{e}}^{-2\gamma t})$$

let's assume $\gamma t \gg 1$ [no transient effects regarded] [choice of time scale in the CoS]

$$\frac{d^2}{dt^2}\langle x^2\rangle = -\gamma\frac{d}{dt}\langle x^2\rangle + \frac{2\pi R_0}{m^2\gamma} = -\gamma\frac{d}{dt}\langle x^2\rangle + \underbrace{\frac{6k_B T}{m}}_{\text{due to energy conservation}}$$

$$\frac{d^2}{dt^2}\langle x^2\rangle + \gamma\frac{d}{dt}\langle x^2\rangle = \frac{6k_B T}{m}$$

to solve this, we need initial conditions

$$\langle x^2(0)\rangle = 0 \quad , \quad \frac{d}{dt}\langle x^2(t)\rangle\Big|_{t=0} = 0$$

$$f'' + \gamma f = c \quad , \quad g = f' \quad , \quad g' = f'' \quad , \quad \boxed{g' + \gamma g = c}$$

$$g(t) = g_0 e^{-\gamma t} + \frac{c}{\gamma}(1 - e^{-\gamma t})$$

now, we use $g = f'$ $\quad f(t) - f(0) = \int_0^t dt'\, g(t')$

we yield $f(t) = f(0) + g(0)\frac{1-e^{-\gamma t}}{\gamma} + \frac{ct}{\gamma} - \frac{c}{\gamma}\frac{1-e^{-\gamma t}}{c}$

but $f(0) = 0 = g(0)$

$$\langle x^2(t)\rangle = \frac{6 k_B T}{m\gamma}\left[t - \underline{\underline{\frac{1}{\gamma}}} + \underline{\underline{\frac{e^{-\gamma t}}{\gamma}}}\right] \quad \| \gamma t \gg 1$$

we have for the time-averaged spatial mean variance $\lim_{t\to\infty}\left[\frac{1}{t}\langle x^2(t)\rangle\right] = \frac{6 k_B T}{m\gamma}\lim_{t\to\infty}\left[1 - \underline{\underline{\frac{1}{\gamma t}}} + \underline{\underline{\frac{e^{-\gamma t}}{t}}}\right]$

$$= \frac{6 k_B T}{m\gamma} \quad \text{(diffusiv)}$$

use green-Kubo formula to yield information for velocity autocorrelation function.

$$\langle \underline{\underline{v(s)}}\, v(0)\rangle = \left\langle \left(v(0)e^{-\gamma s} + \frac{e^{-\gamma s}}{m}\int ds' e^{\gamma s'} R(s')\right) v(0)\right\rangle$$

$$= \langle v_0^2\rangle e^{-\gamma s} + \frac{e^{-\gamma s}}{m}\int_0^s ds'\, e^{\gamma s'} \underbrace{\langle v(0) R(s')\rangle}_{=0}$$

$$\frac{6 k_B T}{m\gamma} = \lim_{t\to\infty}\left[\frac{1}{t}\langle x^2(t)\rangle\right] = 2\int_0^\infty ds \langle v(s) v(0)\rangle$$

$$= 2\int_0^\infty ds\, \langle v_0^2\rangle e^{-\gamma s} = 2\frac{\langle v_0^2\rangle}{-\gamma}[0 - 1] = \frac{2\langle v_0^2\rangle}{\gamma}$$

$\Rightarrow \langle v_0^2\rangle = \frac{3 k_B T}{m}$ to fulfill green-Kubo formula

interestingly, for short times we recover ballistic transport $\langle x^2(t)\rangle\Big|_{\gamma t \ll 1} = \frac{6 k_B T}{m\gamma}\left[t - \frac{1}{\gamma} + \frac{e^{-\gamma t}}{\gamma}\right]$

$$\approx \frac{6 k_B T}{m\gamma}\left[t - \frac{1}{\gamma} + \frac{1 - \gamma t + \frac{1}{2}\gamma t^2}{\gamma}\right]$$

$$= \frac{3 k_B T}{m} t^2 = \langle (\underbrace{v(0) t}_{})^2\rangle = \langle x^2(t)\rangle$$

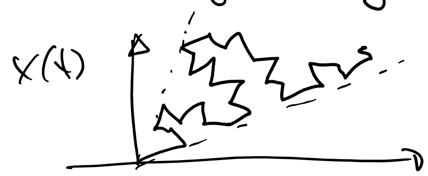
$$= \underbrace{\frac{3 k_B T}{m}}_{\langle v^2(0) \rangle} t^2 = \langle (\underbrace{v(0) t}_{x(t)})^2 \rangle = \langle x^2(t) \rangle$$

<u>note</u>: · the noise function / contribution was essential to ascertain an equilibrium energy distribution in the long time limit and in the average sense

$$\frac{1}{2} m v^2 = U_{KIN} = U_{THERM} = \frac{3}{2} k_B T$$

· noise leads to energy conservation

· the trajectory of the particle is random



nowhere differentiable

$$x(t) = x(0) + \int_0^t dt'\, v(t')$$

$$= x(0) + \frac{v(0)}{\gamma} \left[1 - e^{-\gamma t}\right] + \Delta R(t)$$

<u>but</u>: the spatial mean variance is nevertheless bounded

$$\langle x^2(t) \rangle = \frac{6 k_B T}{m \gamma} \left[t - \frac{1}{\gamma} + \frac{e^{-\gamma t}}{\gamma} \right]$$

· logic has been, we don't know R(t) and therefore we don't know v(t)

we use $U_{KIN} = U_{THERM}$

$$\langle R(t) R(t') \rangle = 6 m k_B T \gamma \, \delta(t - t')$$

· what is it good for?

via the movement of small objects (silica pearls) suspended in liquids reveal characteristic time-scales of the liquid itself, i.e. the theory of Brownian motion is a theory to <u>extract</u> material

[liquid itself, i.e. the theory of Brownian]
motion is a theory to extract material specific parameters (ergodicity, viscosity, memory depth) via observable quantities