4. Quantum Master Equation II.

Dienstag, 16. Oktober 2018 18:48

$$\dot{\rho}_s = -\frac{i}{\hbar}[H_s, \rho_s] - \underbrace{\sum_k |g_k|^2 \delta(\omega_k - \omega_0)}_{= \int d\omega\, J(\omega)} [\bar{n}_k (J\sigma^+\rho_s - \sigma^+\rho_s J + \rho_s \sigma^- J - J\rho_s\sigma^-)$$
$$+ (\bar{n}_k + 1)(J\sigma^-\rho_s - \sigma^-\rho_s J + \rho_s\sigma^+ J - J\rho_s\sigma^+)]$$

$(\sigma^+ + \sigma^-)$

$$\boxed{J = \sigma^+ + \sigma^-}$$

$$= -\frac{i}{\hbar}[H_s, \rho_s] + \int d\omega\, J(\omega) \{\bar{n}(\omega)[\underbrace{2\sigma^+\rho_s\sigma^- - \sigma^-\sigma^+\rho_s - \rho_s\sigma^-\sigma^+}_{= D[\sigma^+]\rho_s} + 2\sigma^+\rho_s\sigma^+]\}$$
$$+ \int d\omega\, J(\omega) \{(\bar{n}(\omega)+1)[\underbrace{2\sigma^-\rho_s\sigma^+ - \sigma^+\sigma^-\rho_s - \rho_s\sigma^+\sigma^-}_{=: D[\sigma^-]\rho_s} + 2\sigma^-\rho_s\sigma^-]\}$$

Discussion: $\cdot\ \Gamma_+ := \int d\omega\, J(\omega)\bar{n}(\omega)$, $\Gamma_- := \int d\omega\, J(\omega)(\bar{n}(\omega)+1)$

- $\Gamma_+ D[\sigma^+]\rho_s$ is induced absorption from the reservoir into the system (gain)
- $\Gamma_- D[\sigma^-]\rho_s$ is spontaneous emission and stimulated emission, contributes even if $n(\omega) = 0$ („true" quantum noise)
- $\sigma^+\rho_s\sigma^+$ and $\sigma^-\rho_s\sigma^-$ vanish typically!

Secular approximation:

to see, they don't matter, we go back into the interaction picture

$$\dot{\rho}_S^I = -\frac{i}{\hbar}[H_S^I(t), \rho_S^I(t)] + \Gamma_+\{D[\sigma^+(t)]\rho_S^I + 2\sigma^+(t)\rho_S\sigma^+(t)\}$$
$$+ \Gamma_-\{D[\sigma^-(t)]\rho_S^I + 2\sigma^-(t)\rho_S\sigma^-(t)\}$$

we know : $\rho_S^I \longrightarrow \exp[\frac{i}{\hbar} H_0^S t]\, \rho_S \exp[-\frac{i}{\hbar} H_0^S t]$

$$\sigma^-(t) = e^{-i\omega_0 t}\sigma^-, \quad \sigma^+(t) = e^{i\omega_0 t}\sigma^+$$

therefore $D[\sigma^+(t)]\rho_S^I(t) =$

$$= e^{-i(\omega_0 - \omega_0)t}(2\sigma^+\rho_S^I\sigma^- - \sigma^-\sigma^+\rho_S^I - \rho_S^I\sigma^-\sigma^+) = D[\sigma^+]\rho_S^I(t)$$

but $2\sigma^+(t)\rho_S^I\sigma^+(t) = e^{i\omega_0 2t}\, 2\sigma^+\rho_S^I(t)\sigma^+$

formally integrate

$t+\Delta t$

$$\rho_S^I(t+\Delta t) - \rho_S^I(t) = -\frac{i}{\hbar}\int_t dt' [H_S^I(t'), \rho_S^I(t')]$$
$$+ \int_t^{t+\Delta t} dt' \, \Gamma_+ \{ D[\sigma^+(t')]\rho_S^I(t') + 2\sigma^+(t')\rho_S^I(t')\sigma^+(t') \}$$
$$+ \Gamma_- \{ \dots \}$$

$$= -\frac{i}{\hbar}\int_t^{t+\Delta t} dt' [H_S^I(t'), \rho_S^I(t')]$$
$$+ \Gamma_+ D[\sigma^+]\int_t^{t+\Delta t} dt' \rho_S^I(t') + \Gamma_+ \sigma^+ \left\{\int_t^{t+\Delta t} dt' e^{i\omega_0 2t'} \rho_S^I(t')\right\}\sigma^+$$
$$+ \Gamma_- \{ \dots \}$$

now assume Δt small so that $\boxed{\rho_S^I(t+\Delta t) \approx \rho_S^I(t)}$ but $\omega_0 \Delta t \gg 1$. however, validity of this assumption strongly depends on system and parameter, e.g. it is not applicable in the ultrastrong coupling limit; however for optical system typically a very safe limit.

$$\rho_S^I(t+\Delta t) - \rho_S^I(t) = -\frac{i}{\hbar}[H_S^I(t), \rho_S^I(t)]\Delta t$$
$$+ \Gamma_+ D[\sigma^+]\rho_S^I(t)\Delta t + \Gamma_+ \sigma^+ \rho_S^I(t)\left\{\int_t^{t+\Delta t} dt' e^{2i\omega_0 t'}\right\}\sigma^+$$
$$+ \Gamma_- \{ \dots \}$$

<u>but</u>: $\int_t^{t+\Delta t} dt' \left(\cos[2\omega_0 t'] + i \sin[2\omega_0 t'] \right) \approx 0, \Delta t \omega_0 \gg 1$

$\Rightarrow$ so we can neglect non-secular terms such as $\sigma^+ \rho_S \sigma^+$ or $\sigma^- \rho_S \sigma^-$
secular approximation is called post trace rotating wave approximation

<u>Summary</u>: $\dot{\rho} = -\frac{i}{\hbar}[H_S + H_B + H_I, \rho]$
with $H_I = \hbar(R J^+ + R^+ J)$, $R = \sum_k g_k r_k$
$[r_k, r_{k'}^+] = \delta(k-k')$

$\longrightarrow$ Born approximation $\rho(t) = \rho_S(t) \otimes \rho_B(t)$
Bath assumption $\rho_B(t) = \rho_B$
Markovian approximation $\rho_S(t-\tau)$

initially uncorrelated states $\rho(0) = \rho_S(0) \otimes \rho_B$

$$\longrightarrow \dot{\rho}_S = -\frac{i}{\hbar}[H_S, \rho_S] + \int_0^t d\tau \Big\{ \langle R^+ R(-\tau)\rangle \big(J^+(-\tau)\rho_S J - J J^+(-\tau)\rho_S\big)$$
$$+ \langle R^+(-\tau) R\rangle \big(J^+\rho_S J(-\tau) - \rho_S J(-\tau) J^+\big)$$
$$+ \langle R R^+(-\tau)\rangle \big(J(-\tau)\rho_S J^+ - J^+ J(-\tau)\rho_S\big)$$
$$+ \langle R(-\tau) R^+\rangle \big(J \rho_S J^+(-\tau) - \rho_S J^+(-\tau) J\big)\Big\}$$

if you want the Lindblad form, you need to apply often secular approximation

<u>two examples</u>: (i) $T=0$, $g_k = \sqrt{\Delta k}\, g_0$ $\quad \dot{\rho}_S = -\frac{i}{\hbar}[H_S, \rho_S] + \frac{\pi g_0^2}{2c} D[J]\rho_S$

choose: $J = \sqrt{2}(\sigma^+ + \sigma^-)$

$$\dot{\rho}_S = -\frac{i}{\hbar}[H_S, \rho_S] + \frac{\pi g_0^2}{c} D[\sigma^+ + \sigma^-]\rho_S$$

use $D[\sigma^+ + \sigma^-]\rho_S = 2(\sigma^+ + \sigma^-)\rho_S(\sigma^- + \sigma^+)$
$$- (\sigma^+ + \sigma^-)^2 \rho_S - \rho_S(\sigma^+ + \sigma^-)^2$$

$$= D[\sigma^-]\rho_S + D[\sigma^+]\rho + \underbrace{2\sigma^+\rho_S\sigma^+}_{\longrightarrow 0} + \underbrace{2\sigma^-\rho_S\sigma^-}_{\longrightarrow 0}$$

$$\dot{\rho}_S = -\frac{i}{\hbar}[H_S, \rho_S] + \frac{\pi g_0^2}{c}\big\{D[\sigma^+]\rho_S + D[\sigma^-]\rho_S\big\}$$

(ii) $T \geq 0$, $\dot{\rho}_S = -\frac{i}{\hbar}[H_S, \rho_S] + \pi \sum_k |g_k|^2 \delta(\omega_k - \omega_0)\bar{n}_k D[\sigma^+]\rho_S$
$$+ \pi \sum_k |g_k|^2 \delta(\omega_k - \omega_0)(\bar{n}_k + 1) D[\sigma^-]\rho_S$$

choose $g_k = \sqrt{\Delta k}\, g_0$ like in (i) use $\sum_k \Delta k \to \int_0^\infty dk$

$$\dot{\rho}_S = -\frac{i}{\hbar}[H_S, \rho_S] + \frac{\pi g_0^2}{c}\big(\bar{n}_{k_0} D[\sigma^+]\rho_S + (\bar{n}_{k_0}+1) D[\sigma^-]\rho_S\big)$$

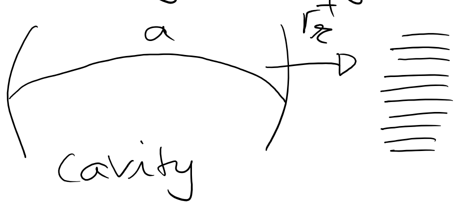
$T \to 0$, $\bar{n}_{k_0} = 0 \implies \dot{\rho}_S = -\frac{i}{\hbar}[H_S, \rho_S] + \frac{\pi g_0^2}{c} D[\sigma^-]\rho_S$

(i) $\dot{\rho}_S\big|_{T=0}$ has also $D[\sigma^+]\rho_S$ which does not appear in (ii)

we see something went wrong in (i), and wrong was to allow non-rotating wave

contribution within the vacuum limit.
in (ii) we applied the second Markovian approximation $\int_0^t d\tau \to \int_0^\infty d\tau$ and then the evaluation of the time integral lead to the $\delta(\omega_k-\omega_0)$. in the first example, due to special circumstance we didn't need to evaluate the time-integral and therefore we allowed negative energies without even knowing it.

easy example: leaky/lossy cavity



reservoir at $T>0$, $n(T)>0$

we are interested in the dynamics of cavity mode intensity $\langle a^\dagger a\rangle$ (observable)

$$\dot\rho = -\frac{i}{\hbar}[H,\rho] + \mathcal{D}[\underbrace{\sqrt{\kappa}\,a}_{=J}]\rho \quad , \quad H = \hbar\omega_0 a^\dagger a$$

$$tr(a^\dagger a\dot\rho) = \frac{d}{dt}\langle a^\dagger a\rangle = \underbrace{tr\left(-\frac{i}{\hbar}[\hbar\omega_0 a^\dagger a,\rho]\,a^\dagger a\right)} + \bar n\kappa\, tr(a^\dagger a\,\mathcal{D}[a^\dagger]\rho) + (\bar n+1)\kappa\, tr(a^\dagger a\,\mathcal{D}[a]\rho)$$

$$= -i\omega_0\underbrace{tr\left(a^\dagger a\rho a^\dagger a - \rho a^\dagger a a^\dagger a\right)}_{=\langle a^\dagger a a^\dagger a\rangle - \langle a^\dagger a a^\dagger a\rangle}$$

$$+(\bar n+1)\kappa\, tr\left(2a^\dagger a\, a\rho a^\dagger - a^\dagger a a^\dagger a\rho - a^\dagger a\rho a^\dagger a\right)$$

$$+\bar n\kappa\, tr\left(2a^\dagger a\, a^\dagger\rho a - a^\dagger a a a^\dagger\rho - a^\dagger a\rho a a^\dagger\right)$$

$$= (\bar n+1)\kappa\left\{2\langle a^\dagger a^\dagger a a\rangle - 2\underbrace{\langle a^\dagger a a^\dagger a\rangle}_{=\langle a^\dagger(\mathbb{1}-a^\dagger a)a\rangle = \langle a^\dagger a\rangle + \langle a^\dagger a^\dagger a a\rangle}\right\}$$

$$+\bar n\kappa\left\{2\langle a a^\dagger a a^\dagger\rangle - \underbrace{\langle a^\dagger a a a^\dagger\rangle}_{\substack{=-\langle a^\dagger a\rangle\\ +\langle a a^\dagger a a^\dagger\rangle}} - \underbrace{\langle a a^\dagger a^\dagger a\rangle}_{\substack{=-\langle a a^\dagger\rangle\\ +\langle a a^\dagger a a^\dagger\rangle}}\right\}$$

$$= -2(\bar n+1)\langle a^\dagger a\rangle + 2\bar n\kappa\underbrace{\langle a a^\dagger\rangle}_{=1+\langle a^\dagger a\rangle}$$

$$\frac{d}{dt}\langle a^\dagger a\rangle = -2\kappa\langle a^\dagger a\rangle + 2\bar{n}\kappa$$

$$\langle a^\dagger a\rangle(t) = e^{-2\kappa t}\langle a^\dagger a\rangle(0) + e^{-2\kappa t}\int_0^t dt'\, 2\bar{n}\kappa e^{2\kappa t'}$$

$$= e^{-2\kappa t}\langle a^\dagger a\rangle(0) + e^{-2\kappa t}\frac{1}{2\kappa}\, 2\bar{n}\kappa\left[e^{2\kappa t} - 1\right]$$

$$= e^{-2\kappa t}\langle a^\dagger a\rangle(0) + \bar{n}(T)\left[1 - e^{-2\kappa t}\right]$$

$$= \bar{n}(T) + e^{-2\kappa t}\left[\langle a^\dagger a\rangle(0) - \bar{n}(T)\right]$$

reasonable result: if $\langle a^\dagger a\rangle(0)$ cavity occupation equals thermal occupation no dynamics at all.
if $\langle a^\dagger a\rangle(0) = 0$, reservoir will pump the cavity
if $\langle a^\dagger a\rangle(0) > \bar{n}(T)$ then we have relaxation

but regardless of $\langle a^\dagger a\rangle(0)$, we will always end up in a steady-state consistent with thermal equilibrium.