5. Liouvillian propagator method.

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goal: derive the Lindblad in the most general without perturbation in the coupling element. this can be done via the Liouvillian Superoperator method.

$$\frac{H}{\hbar} = \underline{\omega_0 c^\dagger c} + \sum_k \underline{\omega_k r_k^\dagger r_k} + \sum_k \left(g_k r_k + g_k^* r_k^\dagger\right) J \qquad \Big|\; \begin{array}{l} J = (c + c^\dagger) \\ J = c^\dagger c \end{array}$$

$\dot{\rho} = -\frac{i}{\hbar}[H, \rho]$ (interaction picture)

$$U(t,0) = \exp\left[-it\left(\omega_0 c^\dagger c + \sum_k \omega_k r_k^\dagger r_k\right)\right]$$

$\dot{\rho}_I = -i\left[B(t) J(t), \rho_I\right]$ with $B(t) = \sum_k \left(g_k r_k e^{-i\omega_k t} + h.c.\right)$

$= \mathcal{L}_I(t)\rho_I(t)$ with $\mathcal{L}_I(t)$ superoperator in Liouville space

what is the Liouville space?

we want a similar equation for the density matrix as we have for the Schrödinger wave function!

Problem: $i\hbar \frac{d}{dt}|\psi\rangle = H|\psi\rangle$ is $i\hbar\frac{d}{dt}\begin{pmatrix}\vdots\end{pmatrix} = \begin{pmatrix}\ddots\end{pmatrix}\begin{pmatrix}\vdots\end{pmatrix}$

but $i\hbar\frac{d}{dt}\rho = [H, \rho]$ is $i\hbar\frac{d}{dt}\begin{pmatrix}\ddots\end{pmatrix} = \left[\begin{pmatrix}\ddots\end{pmatrix}, \begin{pmatrix}\ddots\end{pmatrix}\right]$

trivial approaches to solve the resulting set of differential equations are not possible anymore due to left- or right-sidedness of action

→ let's map the density matrix into a vector.

example: $H = \hbar\Delta|1\rangle\langle 1| + \hbar\Omega\left(|0\rangle\langle 1| + |1\rangle\langle 0|\right)$

$$i\hbar\frac{d}{dt}\rho = \left[\hbar \begin{array}{c|c} & \begin{array}{cc}\langle 0| & \langle 1|\end{array} \\ \begin{array}{c}|0\rangle \\ |1\rangle\end{array} & \begin{pmatrix} 0 & \Omega \\ \Omega & \Delta \end{pmatrix}\end{array}, \begin{pmatrix} c_{00}(t) & c_{01}(t) \\ c_{10}(t) & c_{11}(t) \end{pmatrix}\right]$$

because $\rho = \sum_{ij} c_{ij}(t)|i\rangle\langle j|$

$$\dot{\rho} = -i\begin{pmatrix} c_{10}\Omega - c_{01}\Omega & \Omega(c_{11} - c_{00}) - \Delta c_{01} \\ -\Omega(c_{11} - c_{00}) + \Delta c_{10} & \Omega c_{01} - \Omega c_{10} \end{pmatrix}$$

Liouville space $\rho \longmapsto |\rho\rangle = \sum_{ij} c_{ij}|ij\rangle = \begin{pmatrix} c_{00}(t) \\ c_{01}(t) \\ c_{10}(t) \\ c_{11}(t) \end{pmatrix}$

$$i\hbar|\dot\rho\rangle = \hbar \begin{array}{c|c} & \begin{array}{cccc}\langle 00| & \langle 01| & \langle 10| & \langle 11|\end{array} \\ \begin{array}{c}|00\rangle \\ |01\rangle \\ |10\rangle \\ \end{array} & \begin{pmatrix} 0 & -\Omega & \Omega & 0 \\ -\Omega & -\Delta & 0 & \Omega \\ -\Omega & 0 & \Delta & -\Omega \end{pmatrix}\end{array} \begin{pmatrix} c_{00}(t) \\ c_{01}(t) \\ c_{10}(t) \end{pmatrix}$$

$$i\hbar\,|\rho\rangle = \hbar \begin{array}{c} |00\rangle \\ |01\rangle \\ |10\rangle \\ |11\rangle \end{array} \begin{pmatrix} 0 & -\Omega & \Omega & 0 \\ -\Omega & -\Delta & 0 & \Omega \\ -\Omega & 0 & \Delta & -\Omega \\ 0 & \Omega & -\Omega & 0 \end{pmatrix} \begin{pmatrix} c_{00}(t) \\ c_{01}(t) \\ c_{10}(t) \\ c_{11}(t) \end{pmatrix} \qquad c_{11}(t)$$

we yield basically a Schrödinger equation but for the density matrix $i\hbar \frac{d}{dt}|\rho\rangle = \mathcal{L}\,|\rho\rangle$

integrate formally $|\rho(t)\rangle - |\rho(0)\rangle = \int_0^t dt'\, \mathcal{L}(t')\,|\rho(t')\rangle$

solution via Dyson series

$$|\rho(t)\rangle = |\rho(0)\rangle + \left(-\tfrac{i}{\hbar}\right)\int_0^t dt'\,\mathcal{L}(t')\,|\rho(0)\rangle + \left(-\tfrac{i}{\hbar}\right)^2 \int_0^t dt' \int_0^{t'} dt''\,\mathcal{L}(t')\mathcal{L}(t'')|\rho(0)\rangle + \dots$$

$$= \hat{T}\left\{ \exp\left[-\tfrac{i}{\hbar}\int_0^t dt'\,\mathcal{L}(t')\right] |\rho(0)\rangle \right\}$$

<u>again</u>: without the time-ordering operator it is not a solution of the Liouville-von Neumann equation

$=$ { stop denoting $|\rho\rangle \equiv \rho$ }

$$\rho(t) = \hat{T}\left\{ \exp\left[-\tfrac{i}{\hbar}\int_0^t dt_1\,\mathcal{L}(t_1)\right] \rho(0)\right\} \qquad \mathcal{L}(t_1) = [J_1 B_1, \dots]$$

$$B_1 = B(t_1),\ J_1 = J(t_1)$$

now trace over the bath's degrees of freedom

$$\mathrm{tr}_B\{\rho(t)\} = \rho_S(t) = \mathrm{tr}_B\left\{ \hat{T}\left\{ \exp\left[-\tfrac{i}{\hbar}\int_0^t dt_1\,\mathcal{L}(t_1)\right] \rho(0)\right\}\right\}$$

$$\rho_S(t) = \mathrm{tr}_B\left\{ \hat{T}_{S+B}\left\{ \exp\left[-\tfrac{i}{\hbar}\int_0^t dt_1\,\mathcal{L}(t_1)\right]\right\} \rho(0)\right\}$$

$$= T_S\left\{ \mathrm{tr}_B\left\{ T_B\left\{ \exp\left[-\tfrac{i}{\hbar}\int_0^t dt_1\,\mathcal{L}(t_1)\right]\right\} \rho(0)\right\}\right\}$$

time-reordering on the full superoperator

time-reordering on the reduced superoperator (after the trace over the reservoir)

<u>assume</u>: $\rho(0) = \rho_B \otimes \rho_S(0)$ with $\rho_B = \frac{1}{Z}\exp\left[-\beta \sum_k \hbar\omega_k r_k^\dagger r_k\right]$

for this choice of reservoir odd orders of the Liouvillian vanish identically because we assume thermal equilibrium $\langle r_k\rangle = 0$

$0 = \langle r_k^\dagger r_k r_k\rangle$

we have $\hat{T}_B\left\{ \rho(0) + \frac{1}{2!}\left(-\frac{i}{\hbar}\right)^2 \left[\int_0^t dt_1\,\mathcal{L}(t_1)\right]^2 + \frac{1}{4!}\left(-\frac{i}{\hbar}\right)^4\left[\int_0^t dt_1\,\mathcal{L}(t_1)\right]^4 + \dots\right\}$

$$= \rho(0) - \frac{1}{\hbar^2}\int_0^t dt_1 \int_0^{t_1} dt_2\,\mathcal{L}(t_1)\mathcal{L}(t_2)\rho(0) + \frac{1}{\hbar^4}\int_0^t dt_1 \int_0^{t_1} dt_2 \int_0^{t_2} dt_3 \int_0^{t_3} dt_4\,\mathcal{L}(t_1)\mathcal{L}(t_2)\mathcal{L}(t_3)\mathcal{L}(t_4)\rho(0) + \dots$$

$$T\{A(t_1)B(t_2)\} = A(t_1)B(t_2)\,\Theta(t_1 - t_2) \pm B(t_2)A(t_1)\,\Theta(t_2 - t_1)$$

2nd-order: $\mathcal{L}(t_1)\mathcal{L}(t_2)\rho(0) = [J_1 B_1, [J_2 B_2, \rho(0)]]$

$= [J_1 B_1, J_2 B_2 \rho(0) - \rho(0) J_2 B_2]$

$= J_1 B_1 J_2 B_2 \rho(0) - J_1 B_1 \rho(0) J_2 B_2 - J_2 B_2 \rho(0) J_1 B_1$
$+ \rho(0) J_2 B_2 J_1 B_1$

$$\mathrm{tr}_B\{\mathcal{L}(t_1)\mathcal{L}(t_2)\rho(0)\} = J_1 J_2 \rho_S(0)\langle B_1 B_2\rangle - J_1 \rho_S(0) J_2\, \mathrm{tr}_B\{B_1 \rho_B B_2\}$$
$$- J_2 \rho_S(0) J_1\, \mathrm{tr}_B\{B_2 \rho_B B_1\} + \rho_S(0) J_2 J_1 \langle B_2 B_1\rangle$$

important don't change order of the system's operator

let's write $J_i B_i = J_1 B_1^+ + J_1^+ B_1$ (which includes the case) $J_1 = J_1^+$

$$\mathrm{tr}_B\{\mathcal{L}(t_1)\mathcal{L}(t_2)\rho(0)\} = J_1 J_2^+ \rho_S \langle B_1 B_2^+\rangle - J_1 \rho_S J_2^+ \langle B_2^+ B_1\rangle - J_2^+ \rho_S J_1 \langle B_1^+ B_2\rangle$$
$$+ \rho_S J_2^+ J_1 \langle B_2 B_1^+\rangle + J_1^+ J_2 \rho_S \langle B_1 B_2^+\rangle$$
$$- J_1^+ \rho_S J_2 \langle B_2^+ B_1\rangle - J_2 \rho_S J_1^+ \langle B_1 B_2^+\rangle$$
$$+ \rho_S J_2 J_1^+ \langle B_2^+ B_1\rangle$$

(chapter 3 of Breuer / Petruccione, Feynman-Vernon influence functional: no typos)

now, we choose two examples: (i) $J_i = J_i^+$ (Caldeira-Leggett case)

(ii) $J_i \neq J_i^+$ (quantum optical case)

(i) Brownian motion influence functional

$J_i = J_i^+$

$$\mathrm{tr}_B\{\mathcal{L}(t_1)\mathcal{L}(t_2)\rho(0)\} = \{J_1 J_2 \rho_S - J_2 \rho_S J_1\}\left(\langle B_1^+ B_2\rangle + \langle B_1 B_2^+\rangle\right)$$
$$+ \{\rho_S J_2 J_1 - J_1 \rho_S J_2\}\left(\langle B_2^+ B_1\rangle + \langle B_2 B_1^+\rangle\right)$$

$$\langle B_1 B_2^+\rangle + \langle B_1^+ B_2\rangle = \sum_k \sum_{k'} g_k g_{k'}^* \{ e^{-i\omega_k t_1} e^{i\omega_{k'} t_2} \langle r_k r_{k'}^+\rangle$$
$$+ e^{i\omega_k t_1} e^{-i\omega_{k'} t_2} \underbrace{\langle r_k^+ r_{k'}\rangle}_{= n_k \delta(k-k')} \}$$

$$= \sum_k |g_k|^2 \left\{ e^{-i\omega_k(t_1-t_2)}(n_k+1) + n_k e^{i\omega_k(t_1-t_2)} \right\}$$

$$= \sum_k |g_k|^2 \left\{ e^{-i\omega_k(t_1-t_2)}(n_k+\tfrac{1}{2}) + \tfrac{1}{2} e^{-i\omega_k(t_1-t_2)} + e^{i\omega_k(t_1-t_2)}(n_k+\tfrac{1}{2}) - \tfrac{1}{2} e^{i\omega_k(t_1-t_2)} \right\}$$

$$= \sum_k |g_k|^2 \left\{ 2\cos[\omega_k(t_1-t_2)](n_k+\tfrac{1}{2}) + i \sin[-\omega_k(t_1-t_2)] \right\}$$

$$= \sum_k |g_k|^2 \left\{ \cos[\omega_k(t_1-t_2)](2n_k+1) - i \sin[\omega_k(t_1-t_2)] \right\}$$

$$\langle B_2 B_1^+ \rangle + \langle B_2^+ B_1 \rangle = \sum_k |g_k|^2 \left\{ \cos[\omega_k(t_1-t_2)](2n_k+1) + i \sin[\omega_k(t_1-t_2)] \right\}$$

$$D_c(t_1-t_2) = \sum_k |g_k|^2 \cos[\omega_k(t_1-t_2)](2n_k+1)$$

$$= \sum_k |g_k|^2 \cos[\omega_k(t_1-t_2)] \coth\left[\frac{\hbar\omega_k}{2k_BT}\right]$$

$$D_s(t_1-t_2) = \sum_k |g_k|^2 \sin[\omega_k(t_1-t_2)]$$

$$\mathrm{tr}_B\{\mathcal{L}(t_1)\mathcal{L}(t_2)\rho(0)\} = \{J_1J_2\rho_s - J_2\rho_sJ_1\}[D_c(t_1-t_2) - iD_s(t_1-t_2)]$$
$$+ \{\rho_sJ_2J_1 - J_1\rho_sJ_2\}[D_c(t_1-t_2) + iD_s(t_1-t_2)]$$

$$= D_c(t_1-t_2)\,[J_1,[J_2,\rho_s]]$$
$$- iD_s(t_1-t_2)\,[J_1,\{J_2,\rho_s\}]$$

hence, $\rho_s(t) = \hat{T}_s\left\{ \exp\left[-\frac{1}{\hbar^2}\int_0^t dt_1 \int_0^{t_1} dt_2 \{D_c(t_1-t_2)[J_1,[J_2,\dots]] - iD_s(t_1-t_2)[J_1,\{J_2,\dots\}]\}\right]\rho_s(0)\right\}$

So, this is the very famous Feyman-Vernon influence functional.

the $[J,[J,\dots]]$ allows a Lindblad formulation but $[J,\{J,\dots\}]$ will never lead to Lindblad physics.

first example of an exactly solvable non-Markovian model.

(ii) $J_i \neq J_i^+$ (Lindblad case)

(1.) rotating wave approximation
(2.) $T=0$
(3.) initially a product state $\rho(t) = \rho_s(0) \otimes \rho_B$
(4.) $H_{sys} = \hbar\omega_0 c^+ c$
(5.) $g_k = \hbar\sqrt{\frac{\Delta k \, \delta^2}{\pi}} = \text{const}$

$$\mathrm{tr}_B\{\mathcal{L}(t_1)\mathcal{L}(t_2)\rho(0)\}\big|_{vac} = \{\text{only } \langle BB^+\rangle \text{ contribute}\}$$

$$= J_1^+ J_2 \rho_s \langle B_1 B_2^+\rangle + \rho_s J_2^+ J_1 \langle B_2 B_1^+\rangle - J_2\rho_s J_1 \langle B_1 B_2^+\rangle$$
$$- J_1 \rho_s J_2^+ \langle B_2 B_1^+\rangle$$

we know $\frac{1}{\hbar^2}\langle B_i B_j^+\rangle = \sum_{k k'} g_k g_{k'}^* e^{-i\omega_k t_i + i\omega_{k'} t_j} \langle r_k r_{k'}^+\rangle$

$$= \sum_k |g_k|^2 e^{-i\omega_k(t_i - t_j)} = \frac{c\gamma}{\pi} \sum_k \Delta k \, e^{-i\omega_k(t_i-t_j)} = \gamma\, \delta(t_i - t_j)$$

$$= \langle B_j B_i^+ \rangle$$

$$\mathrm{tr}_B\{ \mathcal{L}(t_1)\mathcal{L}(t_2)\rho(0)\} = -\gamma\hbar^2 \delta(t_1-t_2)\{ J_1 \rho_S J_2^+ + J_2 \rho_S J_1^+ - J_1^+ J_2 \rho_S - \rho_S J_2^+ J_1 \}$$

hence, $\rho_S(t) = T_S\{ \exp[\gamma \int_0^t dt_1 \int_0^{t_1} dt_2\, \delta(t_1-t_2)\{\ldots\}]\rho_S(0)\}$

$$= T_S\{\exp[\gamma \int_0^t dt_1 \{2 J_1 \boxtimes J_1^+ - J_1^+ J_1 \boxtimes 1 - \boxtimes J_1^+ J_1\}]\rho_S(0)\}$$

$H_{sys} = \hbar\omega_0 c^\dagger c$, $J(t_1) = e^{-i\omega_0 t_1} J$

(special choice of H_{sys})

$$\rho_S(t) = T_S\{\exp[\gamma t\, D[J]]\rho_S(0)\}$$, now $T_S \equiv \underline{1\!\!1}$

$$= e^{\gamma t D[J]} \rho_S(0)$$

$$\frac{d}{dt}\rho_S(t) = \underbrace{e^{\gamma t D[J]} \gamma D[J]}_{\text{commutes}} \rho_S(0) = \gamma D[J]\, e^{\gamma t D[J]} \rho_S(0)$$

$$= \gamma D[J] \rho_S(t)$$

what about fourth-order?

$$\mathrm{tr}_B\{\mathcal{L}(t_1)\mathcal{L}(t_2)\mathcal{L}(t_3)\mathcal{L}(t_4)\rho(0)\}$$

$$= \mathrm{tr}_B\{ [J_1 B_1^+ + J_1^+ B_1, [J_2 B_2^+ + J_2^+ B_2, [J_3 B_3^+ + J_3^+ B_3, [J_4 B_4^+ + J_4^+ B_4, \rho(0)]]]]\}$$

a huge amount of terms but not all contribute.
first, we assume the vacuum: only non-normal ordered operator sequences
now, we use the time-order. what can happen?

$\delta(t_1-t_2)\delta(t_3-t_4)$, $\delta(t_1-t_3)\delta(t_2-t_4)$, $\delta(t_1-t_4)\delta(t_2-t_3)$

$\Rightarrow$ three possible ways the time-ordered time integrals can be evaluated

$$\int_0^t dt_1 \int_0^{t_1} dt_2 \int_0^{t_2} dt_3 \int_0^{t_3} dt_4 = \int_0^t dt_1 \int_0^\infty dt_2\, \Theta(t_1-t_2) \int_0^\infty dt_3\, \Theta(t_2-t_3) \int_0^\infty dt_4\, \Theta(t_3-t_4)$$

(i) $\delta(t_1-t_2)\delta(t_3-t_4) \longrightarrow \int_0^t dt_1 \underbrace{\Theta(t_1-t_1)}_{=1} \int_0^\infty dt_3\, \Theta(t_1-t_3) \underbrace{\Theta(t_3-t_3)}_{=1}$

$$= \int_0^t dt_1 \int_0^{t_1} dt_3 = \frac{1}{2}t^2$$

$$(ii)\quad \delta(t_1-t_3)\,\delta(t_2-t_4) \longrightarrow \int_0^t dt_1 \int_0^\infty dt_2\; \Theta(t_1-t_2)\,\Theta(t_2-t_1)\,\Theta(t_1-t_1) = \int_0^t dt_1 \int_{t_1}^{t_1} dt_2 = 0$$

$$(iii)\quad \delta(t_1-t_4)\,\delta(t_2-t_3) \longrightarrow \int_0^t dt_1 \int_0^\infty dt_2\; \Theta(t_1-t_2)\,\Theta(t_2-t_2)\,\Theta(t_2-t_1) = 0$$

only terms contribute which yield $\delta(t_1-t_2)\,\delta(t_3-t_4)$

examples $B_1 B_2^\dagger B_3 B_4$, or $B_4 B_1 B_2^\dagger B_3^\dagger$ ...

still many contributions, nevertheless one yields

$$\mathrm{tr}_B\{\mathcal{L}(t_1)\mathcal{L}(t_2)\mathcal{L}(t_3)\mathcal{L}(t_4)\rho(0)\} = \frac{1}{2}\gamma^2 t^2 D[J]\{D[J]\rho_S(0)\}$$
$$= \frac{1}{2!}\left(\gamma t\, D[J]\right)^2 \rho_S(0)$$

$$\mathrm{tr}_B\{\mathcal{L}(t_1)\mathcal{L}(t_2)\mathcal{L}(t_3)\mathcal{L}(t_4)\rho(0)\}\Big|_{\text{vacuum}} =$$

$$\begin{aligned}
&= J_1 J_2^\dagger J_3 J_4^\dagger \rho_S^{(\omega)} \langle B_1 B_2^\dagger B_3 B_4^\dagger\rangle - J_1 J_2^\dagger J_3^\dagger \rho_S^{(\omega)} J_4 \langle B_4 B_1 B_2^\dagger B_3^\dagger\rangle - J_1 J_2^\dagger J_4^\dagger \rho J_3 \langle B_3 B_1 B_2^\dagger B_4^\dagger\rangle + J_1 J_2^\dagger \rho_S^{(\omega)} J_4 J_3^\dagger \langle B_4 B_3^\dagger B_1 B_2^\dagger\rangle \\
&+ J_1 J_3^\dagger \rho J_4 J_2^\dagger \langle B_4 B_2^\dagger B_1 B_3^\dagger\rangle + J_1 J_4^\dagger \rho J_3 J_2^\dagger \langle B_3 B_2^\dagger B_1 B_4^\dagger\rangle + J_1 J_2 J_3^\dagger J_4^\dagger \rho \langle B_1 B_2 B_3^\dagger B_4^\dagger\rangle - J_1 J_3^\dagger J_4^\dagger \rho J_2 \langle B_2 B_1 B_3^\dagger B_4^\dagger\rangle \\
&- J_2^\dagger J_3 J_4^\dagger \rho J_1 \langle B_1 B_2^\dagger B_3 B_4^\dagger\rangle + J_2^\dagger J_3^\dagger \rho J_4 J_1 \langle B_4 B_1 B_2^\dagger B_3^\dagger\rangle + J_2^\dagger J_4^\dagger \rho J_3 J_1 \langle B_3 B_1 B_2^\dagger B_4^\dagger\rangle - J_2^\dagger \rho J_4 J_3^\dagger J_1 \langle B_4 B_3^\dagger B_1 B_2^\dagger\rangle \\
&- J_3^\dagger \rho J_4 J_2^\dagger J_1 \langle B_4 B_2^\dagger B_1 B_3^\dagger\rangle - J_4^\dagger \rho J_3 J_2^\dagger J_1 \langle B_3 B_2^\dagger B_1 B_4^\dagger\rangle - J_2 J_3^\dagger J_4^\dagger \rho J_1 \langle B_1 B_2 B_3^\dagger B_4^\dagger\rangle + J_3^\dagger J_4^\dagger \rho J_2 J_1 \langle B_2 B_1 B_3^\dagger B_4^\dagger\rangle \\
&+ J_1^\dagger J_2^\dagger \rho J_4 J_3 \langle B_4 B_3 B_1^\dagger B_2^\dagger\rangle - J_1^\dagger \rho J_4 J_3 J_2^\dagger \langle B_4 B_3 B_2^\dagger B_1^\dagger\rangle - J_1^\dagger J_2 J_3^\dagger \rho J_4 \langle B_4 B_1^\dagger B_2 B_3^\dagger\rangle \\
&- J_1^\dagger J_2 J_4^\dagger \rho J_3 \langle B_3 B_1^\dagger B_2 B_4^\dagger\rangle - J_1^\dagger J_3 J_4^\dagger \rho J_2 \langle B_2 B_1^\dagger B_3 B_4^\dagger\rangle + J_1^\dagger J_3^\dagger \rho J_4 J_2 \langle B_4 B_2 B_1^\dagger B_3^\dagger\rangle + J_1^\dagger J_4^\dagger \rho J_3 J_1 \langle B_3 B_2 B_1^\dagger B_4\rangle \\
&- J_1^\dagger \rho J_4 J_3^\dagger J_2 \langle B_4 B_3^\dagger B_2 B_1^\dagger\rangle - J_2^\dagger \rho J_4 J_3 J_1^\dagger \langle B_4 B_3 B_1^\dagger B_2^\dagger\rangle + \rho J_4 J_3 J_2^\dagger J_1^\dagger \langle B_4 B_3 B_2^\dagger B_1^\dagger\rangle + J_2 J_3^\dagger \rho J_4 J_1^\dagger \langle B_4 B_1^\dagger B_2 B_3^\dagger\rangle \\
&+ J_2 J_4^\dagger \rho J_3 J_1^\dagger \langle B_3 B_1^\dagger B_2 B_4^\dagger\rangle + J_3 J_4^\dagger \rho J_2 J_1^\dagger \langle B_2 B_1^\dagger B_3 B_4^\dagger\rangle - J_3^\dagger \rho J_4 J_2 J_1^\dagger \langle B_4 B_2 B_1^\dagger B_3^\dagger\rangle - J_4^\dagger \rho J_3 J_2 J_1^\dagger \langle B_3 B_2 B_1^\dagger B_4^\dagger\rangle \\
&+ \rho J_4 J_3^\dagger J_2 J_1^\dagger \langle B_4 B_3^\dagger B_2 B_1^\dagger\rangle
\end{aligned}$$

$$\begin{aligned}
&= \hbar^4 \gamma^2 \delta(t_1-t_2)\,\delta(t_3-t_4)\Big\{ 4 J_1^\dagger J_3^\dagger \rho J_3 J_1 + 2 J_3 J_3^\dagger \rho J_1 J_1^\dagger + \rho J_3 J_3^\dagger J_1 J_1^\dagger + J_1 J_1^\dagger J_3 J_3^\dagger \rho \\
&\qquad - 2 J_1^\dagger J_3 J_3^\dagger \rho J_1 - 2 J_1^\dagger \rho J_3 J_3^\dagger J_1 - 2 J_3^\dagger \rho J_3 J_1 J_1^\dagger - 2 J_1 J_1^\dagger J_3^\dagger \rho J_3 \Big\}
\end{aligned}$$

<u>note</u>:
- we have derived the Lindblad master equation via perturbation theory and exactly via the propagator method.
- the propagator method allows to go beyond the Markovian version of the master equation
- so, we can also derive non-Markovian but trace-preserving master equations

via time-dependent damping $\Gamma(t)$.