7. Quantum Optics. Mollow model.

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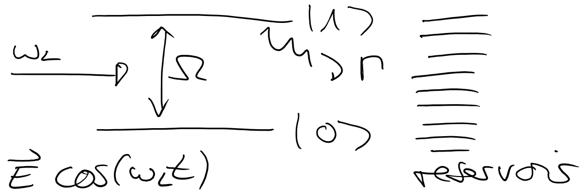
Until now, the system's dynamics was assumed to be free, mainly governed by H_0. We will now discuss a <u>driven</u>-dissipative system: the Mollow model.

Mollow model one of the main paradigm of quantum optics



$\vec{E} = \vec{E}\cos(\omega_L t)$

$\frac{d \cdot E_0}{\hbar} \equiv$ Rabi frequency

$$H/\hbar = \omega_0 \sigma_{11} + \Omega[\sigma_{01} + \sigma_{10}] + \sum_k \omega_k r_k^\dagger r_k + \sum_k g_k\left(r_k^\dagger \sigma_{01} + \sigma_{10} r_k\right)$$

when is our model Markovian? $\overline{n}_k = 0$ (vacuum case) and what we also is $g_k \cong \text{const} = g_0$.

(1.) for a photon reservoir the vacuum state is a very good approximation. typically optically active transitions are in the range of eV (e.g. 2eV)
let's take the Bose-Einstein-Distribution

$$n_k = \left[\exp\left[\frac{\hbar\omega_k}{k_B T}\right] - 1\right]^{-1} = \left\{\hbar\omega_k = 2eV,\ k_B = 8.61 \cdot 10^{-5}\frac{eV}{K}\right\}$$

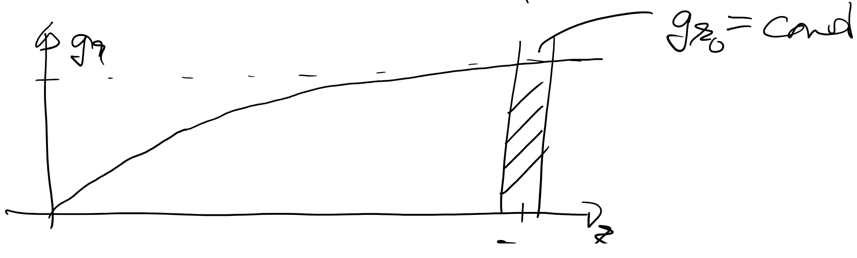
$$\approx \left[\exp\left[\frac{10^4 K}{T}\right] - 1\right]^{-1}$$

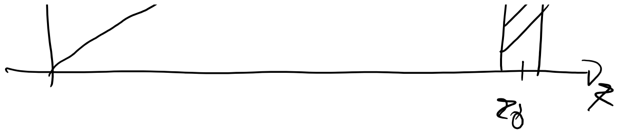
so even at room temperature (300K), n_k is a very small number and therefore negligible
[beware: not valid for microwave quantum optics (Haroche)]

$$\overline{n}_k = 0, \quad \rho_B = |vac\rangle\langle vac|$$

(2.) the electron-photon is determined via the dipole moment times the vacuum field amplitude: $\mathcal{E}_{vac}^k = \sqrt{\frac{\hbar\omega_k}{2V}}$, therefore $g_k \neq$ const.

$$\frac{d}{dk} k^{1/2} = (2\sqrt{k})^{-1} \longrightarrow 0 \text{ for large } k$$





so quantum optics are mainly governed by Markovian dissipation as long as no others reservoirs present (phonons)

interaction picture $H_I = g_0 \sum_z \{ r_z^+(t)\, \sigma^-(t) + \sigma_+(t)\, r_z(t) \}$

(until now: $\sigma^-(t) = \sigma^- e^{-i\omega_0 t}$ free evolution)

but now, we have a more involved system dynamics

$$\sigma^-(t) = \underbrace{\exp\left[it \left[\omega_0 \sigma_{11} + \Omega(\sigma^- + \sigma^+) \right] \right]}_{= U^+(t,0)} \sigma^- \; U(t,0)$$

$$\sigma^-(t) = e^{-i\omega_0 t} \Big[\sigma^-(0) + \tfrac{1}{2}(1 - \cos\Omega t)\,(\sigma^+(0) - \sigma^-(0)) - \tfrac{1}{2}\, i \sin\Omega t \; \sigma_z(0) \Big]$$

$$r_z(t) = e^{-i\omega_z t}\, r_z(0)$$

this is valid $\Omega(t) \equiv$ const.
resonance $\omega_L = \omega_0$

goal: derive the Mollow master equation.

$$\rho_I^S(t) = T_S \Big\{ \mathrm{tr}_B \Big\{ T_B \Big\{ \exp\Big[\int_0^t dt_1\, \mathcal{L}(t_1) \Big] \rho_\omega(0) \Big\} \Big\} \Big\}$$

$$= T_S \Big\{ \rho_\omega^S(0) - \frac{1}{\hbar^2} \int_0^t dt_1 \int_0^{t_1} dt_2\; \mathrm{tr}_B \{ \mathcal{L}(t_1)\, \mathcal{L}(t_2)\, \rho_\omega(0) \} + \dots \Big\}$$

the environment is bosonic: $\sum_z \sum_{z'} |g_0|^2 \langle r_z(t)\, r_{z'}^+(t') \rangle = \hbar^2 \gamma\, \delta(t - t')$

$$\mathrm{tr}_B \{ \mathcal{L}(t_1)\mathcal{L}(t_2)\, \rho_I(0) \} = \int_0^t dt_1\, \hbar^2 \gamma \, \{ 2\sigma^-(t_1)\, \rho_I^S(0)\, \sigma^+(t_1) - \sigma^+(t_1)\, \sigma^-(t_1)\, \rho_I^S(0) \; \rho_I^S(0)\, \sigma^+(t_1)\, \sigma^-(t_1) \}$$

$$\sigma^-(t_1) = e^{-i\omega_0 t_1} \left[(1 - f(t_1))\, \sigma^- + f(t_1)\, \sigma^+ + g(t_1)\, \sigma_z \right]$$

define:
$f_1(t) = \tfrac{1}{2}(1 + \cos\Omega t)$, $\sigma_1 \equiv \sigma^-$
$f_2(t) = \tfrac{1}{2}(1 - \cos\Omega t)$, $\sigma_2 \equiv \sigma^+$
$f_3(t) = -\tfrac{i}{2}\sin\Omega t$, $\sigma_3 \equiv \sigma_z$

we can write

$$\int_0^t dt_1\, D[J(t_1)] = \int_0^t dt_1\, D\Big[\sum_{i=1}^{3} f_i(t_1)\sigma_i\Big]$$

$$= \int_0^t dt_1 \sum_{i=1}^{3}\sum_{j=1}^{3} f_i(t_1) f_j^*(t_1)\, \{2\sigma_i \boxtimes \sigma_j - \{\sigma_j\sigma_i, \boxtimes\}\}$$

$$= \sum_{i,j=1}^{3} F_{ij}(t)\, \{2\sigma_i \boxtimes \sigma_j - \{\sigma_j\sigma_i, \boxtimes\}\} \qquad F_{ij}(t) = \int_0^t dt' f_i f_j^*$$

we can find the corresponding master equation

$$\rho_I^S(t) = \underbrace{T_S}_{=\mathbb{1}}\Big\{\exp\Big[\gamma \sum_{i,j=1}^{3} F_{ij}(t)\{2\sigma_i\boxtimes\sigma_j - \{\sigma_j\sigma_i, \boxtimes\}\}\Big]\rho_I^S(0)\Big\}$$

$$\frac{d}{dt}\rho_I^S(t) = \exp\Big[\gamma\sum_{i,j=1}\{\dots\}\Big]\,\gamma\sum_{i,j=1}^{3}\dot F_{ij}\,\{2\sigma_i\boxtimes\sigma_j - \{\sigma_j\sigma_i,\boxtimes\}\}\rho_I^S(0)$$

$$= \gamma\sum_{ij}\dot F_{ij}(t)\,\{2\sigma_i\rho_I^S(t)\sigma_j - \sigma_j\sigma_i\rho_I^S(t) - \rho_I^S(t)\sigma_j\sigma_i\}$$

$$= \gamma\,\{2\sigma^-(t)\rho_I^S(t)\sigma^+(t) - \sigma^+(t)\sigma^-(t)\rho_I^S(t) - \rho_I^S\sigma^+(t)\sigma^-(t)\}$$

$$\boxed{\frac{d}{dt}\int_0^t dt'\, g_1(t')g_2^*(t') = g_1(t)g_2^*(t)}$$

$$\frac{d}{dt}\rho_S = -\frac{i}{\hbar}[\Omega(\sigma^+ + \sigma^-), \rho_S] + \gamma D[\sigma^-]\rho_S$$

$\hat{=}$ the quantum master equation is <u>exact</u> for our case (optically driven atoms, e.g.); the system's dynamics does <u>not</u> change the dissipative properties because of the structureless reservoir coupling $g_k \equiv g_{k_0} = \text{const}$!!

check with quantum Langevin: (Heisenberg picture)

$$\dot\sigma_{11} = \frac{i}{\hbar}[H, \sigma_{11}] = i\Omega[\sigma_{01}+\sigma_{10}, \sigma_{11}] + i\sum_k g_0[r_k\sigma_{10} + r_k^\dagger\sigma_{01}, \sigma_{11}]$$

$$= i\Omega(\sigma_{01} - \sigma_{10}) + i\sum_k g_0[r_k^\dagger\sigma_{01} - \sigma_{10}r_k]$$ (attention: keep normal-ordering)

$$\dot r_k = -i\omega_k r_k - i g_0\sigma_{01}, \quad r_k(t) = e^{-i\omega_k t} r_k(0) - i\int_0^t dt'\, e^{-i\omega_k(t-t')}\sigma_{01}(t')$$

$$\dot\sigma_{11} = i\Omega(\sigma_{01}-\sigma_{10}) + i g_0 \sum_k \Big\{ e^{i\omega_k t} r_k^\dagger(0)\,\sigma_{01}(t) + i g_0 \int_0^t dt'\, e^{i\omega_k(t-t')}\sigma_{10}(t')\,\sigma_{01}(t) \Big\}$$

$$- i g_0 \sum_k \Big\{ e^{-i\omega_k t} r_k(0)\,\sigma_{10}(t) - i \cdots \Big\}$$

$$\dot\sigma_{11} = i\Omega(\sigma_{01}-\sigma_{10}) + i\Delta R^\dagger(t)\,\sigma_{01}(t) - \int_0^t dt'\,\sigma_{10}(t') \sum_k g_0^2 e^{i\omega_k(t-t')}\,\sigma_{01}(t)$$
$$- i\sigma_{10}(t)\,\Delta R(t) - \dots$$

$$\dot\sigma_{11} = i\Omega(\sigma_{01}-\sigma_{01}) + i\Delta R^\dagger(t)\,\sigma_{01}(t) - \Gamma\sigma_{11}(t)$$
$$- i\sigma_{10}(t)\,\Delta R(t) - \Gamma\sigma_{11}(t)$$

$$\langle 1,vac|\dot\sigma_{11}|1,vac\rangle = \langle\dot\sigma_{11}(t)\rangle = i\Omega\big(\langle\sigma_{01}\rangle - \langle\sigma_{10}\rangle\big) - 2\Gamma\langle\sigma_{11}(t)\rangle$$
$$+ i\,\underbrace{\langle 1,vac|\Delta R^\dagger(t)}_{=0}\,\sigma_{01}|1,vac\rangle$$
$$- i\,\langle 1,vac|\,\sigma_{10}(t)\,\underbrace{\Delta R(t)|1,vac\rangle}_{=0}$$

$$\dot\sigma_{10} = i\Omega\,[\mathbb{1} - 2\sigma_{11}(t)] + i\Delta R^\dagger(t)\,[\mathbb{1} - 2\sigma_{11}(t)]$$
$$- \Gamma\sigma_{10}(t)$$

$$\langle\dot\sigma_{10}\rangle = i\Omega\,[1 - 2\sigma_{11}(t)] - \Gamma\langle\sigma_{10}(t)\rangle$$
$$\uparrow$$
$$\langle\sigma_{00}(t)\rangle - \langle\sigma_{11}(t)\rangle = \{\langle\sigma_{00}(t)\rangle + \langle\sigma_{11}(t)\rangle = 1\}$$

solving the Bloch equations:

$$\dot\sigma_{11} = -2\Gamma_R\sigma_{11} - i\Omega_L[\sigma_{01}-\sigma_{10}]$$
$$\dot\sigma_{00} = 2\Gamma_R\sigma_{11} + i\Omega_L[\sigma_{01}-\sigma_{10}]$$
$$\dot\sigma_{01} = -\Gamma_R\sigma_{01} - i\Omega_L[\sigma_{11}-\sigma_{00}]$$
$$\dot\sigma_{10} = -\Gamma_R\sigma_{10} + i\Omega_L[\sigma_{11}-\sigma_{00}]$$

we introduce now convenient variables

$$W = \langle\sigma_{11}-\sigma_{00}\rangle, \quad B = \langle\sigma_{01}-\sigma_{10}\rangle$$
$$\Sigma = \langle\sigma_{11}+\sigma_{00}\rangle \quad D = \langle\sigma_{01}+\sigma_{10}\rangle$$

$$\dot W = -4\Gamma_R\langle\sigma_{11}\rangle - i2\Omega_L B(t)$$
$$= -2\Gamma_R W(t) - 2\Gamma_R\Sigma - i2\Omega_L B(t)$$

$$\dot{B} = -\Gamma_R B(t) - i\, 2\Omega_L W(t)$$

solve via the Laplace transform: $\Omega = 2\Omega_L$, $\Gamma = 2\Gamma_R$

$$sW(s) - W(0) = -\Gamma W(s) - \Gamma \Sigma \frac{1}{s} - i\Omega B(s)$$

$$sB(s) - B(0) = -\frac{\Gamma}{2} B(s) - i\Omega W(s)$$

$$B(s) = \frac{1}{s+\Gamma/2}\left[B(0) - i\Omega W(s)\right]$$

$$sW(s) = W(0) - \Gamma W(s) - \frac{\Gamma}{s}\Sigma - i\Omega \frac{B(0) - i\Omega W(s)}{s+\Gamma/2}$$

$$W(s) = \frac{s+\Gamma/2}{\underbrace{(s+\Gamma)(s+\Gamma/2)+\Omega^2}} \left[W(0) - \frac{\Gamma}{s}\Sigma - i\frac{B(0)}{s+\Gamma/2}\right]$$

$$= s^2 + \frac{3}{2}\Gamma s + \frac{\Gamma^2}{2} + \Omega^2 = \left(s+\frac{3\Gamma}{4}\right)^2 + \Omega^2 - \frac{\Gamma^2}{16}$$

$$W(s) = W(0)\frac{s+\Gamma/2}{(s+\frac{3\Gamma}{4})^2 + \Omega^2 - \frac{\Gamma^2}{16}} - \Sigma(0)\left(\Gamma \frac{1}{(s+\frac{3}{4}\Gamma)^2 + \Omega^2 - \frac{\Gamma^2}{16}} + \frac{\Gamma^2}{2s}\frac{1}{(s+\frac{3}{4}\Gamma)^2 + \Omega^2 - \frac{\Gamma^2}{16}}\right)$$

$$+ B(0)\frac{-i\Omega}{(s+\frac{3}{4}\Gamma)^2 + \Omega^2 - \frac{\Gamma^2}{16}}$$

transform back into the time domain

$$\mathcal{L}\left[\frac{as^2+bs+c}{[(s+d)^2+e^2](s+f)}\right] = (a-A)e^{-dt}\cos(et) + $$
$$+ A e^{-ft}$$
$$+ \frac{1}{e}\left[b - a(d+f) - A(d-f)\right]e^{-dt}\sin et$$

$$A := \frac{af^2 - bf + c}{(d-f)^2 + e^2}$$

$$\mathcal{L}\left[\frac{as+b}{(s+c)^2+d^2}\right] = a e^{-ct}\cos(dt) + \frac{b-ca}{d}e^{-ct}\sin(dt)$$

$$\bar\Omega = \sqrt{\Omega^2 - \Gamma^2/16}, \quad \bar\Gamma = \frac{3\Gamma}{4}$$

$$W(t) = -\Sigma(0)\left[\frac{\Gamma}{\bar\Omega}e^{-\bar\Gamma t}\sin\bar\Omega t - \frac{\Gamma^2}{2\Omega^2+\Gamma^2}\left(e^{-\bar\Gamma t}\cos\bar\Omega t - 1 + \frac{\bar\Gamma}{\bar\Omega}e^{-\bar\Gamma t}\sin\bar\Omega t\right)\right]$$

[illegible]

$$W(t) = -\Sigma(0)\Big[\frac{\cdot}{\bar\Omega} e \;\; \sin\bar\Omega t - \frac{\cdot}{2\Omega^2+\Gamma^2}\big(e \;\; \cos\bar\Omega \;\cdots\; \Omega \;\cdots\big)\Big]$$

$$+W(0)\left[e^{-\bar\Gamma t}\cos\bar\Omega t - \frac{\Gamma}{4\bar\Omega} e^{-\bar\Gamma t}\sin\bar\Omega t\right]$$

$$+B(0)\; \frac{-i\Omega}{\bar\Omega}\, e^{-\bar\Gamma t}\sin\bar\Omega t$$

$$W(\infty) = -\Sigma(0)\frac{\Gamma^2/2}{\Omega^2+\bar\Gamma^2} = -\Sigma(0)\frac{\Gamma^2/2}{\Omega^2-\frac{\Gamma^2}{16}+\frac{9\Gamma^2}{16}} = -\Sigma(0)\frac{1}{\frac{2\Omega^2}{\Gamma^2}+1}$$

$$\Omega=0 \quad W(\infty) = -1 \quad \text{if } \Sigma(0)=1$$

$$W(\infty) = \langle\sigma_{11}(\infty)\rangle - \langle\sigma_{00}(\infty)\rangle$$

no excitation limit : $\Omega_L = 0$, $\bar\Omega = \sqrt{0^2-\frac{\Gamma^2}{16}} = i\frac{\Gamma}{4}$

$$W(t)\Big|_{\Omega_L=0} = -\Sigma(0) + \Sigma(0)\, i\, e^{-\bar\Gamma t}\sin\left[i\tfrac{\Gamma}{4}t\right]$$
$$+\Sigma(0)\, e^{-\bar\Gamma t}\cos\left[i\tfrac{\Gamma}{4}t\right]$$
$$+W(0)\left\{e^{-\bar\Gamma t}\cos\left[i\tfrac{\Gamma}{4}t\right] - \frac{1}{i}e^{-\bar\Gamma t}\sin\left[i\tfrac{\Gamma}{4}t\right]\right\}$$

$$= -\Sigma(0) + \left[\langle\sigma_{00}(0)+\sigma_{11}(0)+\sigma_{11}(0)-\sigma_{00}(0)\rangle\right]e^{-\bar\Gamma t}e^{-\frac{\Gamma}{4}t}$$

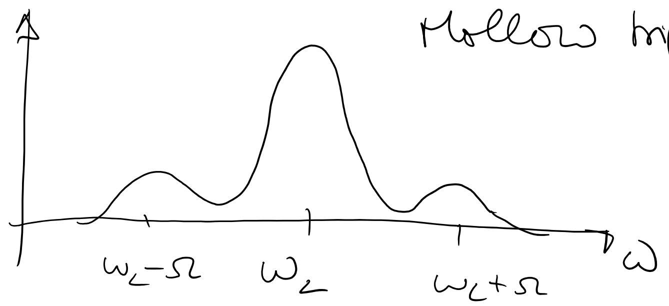
$$= -\Sigma(0) + 2\langle\sigma_{11}(0)\rangle e^{-\Gamma t}$$

this is the correct result : $\langle\sigma_{11}(t)\rangle = \dfrac{W(t)+\Sigma(0)}{2}$

$$\langle\sigma_{11}(t)\rangle = W(\infty)+\frac{1}{2} = \frac{\Omega^2-\frac{\Gamma^2}{2}}{2\Omega^2+\Gamma^2} \xrightarrow{\Omega/\Gamma \gg 1} \frac{1}{2}$$

$$S(\omega) = \mathrm{Re}\left[\int_0^\infty d\tau\, \langle\sigma^+(0)\sigma^-(\tau)\rangle e^{-i\omega\tau}\right]$$

$S(\omega)$



Mollow triplet