# 8. Non-Markovian Quantum Optics.

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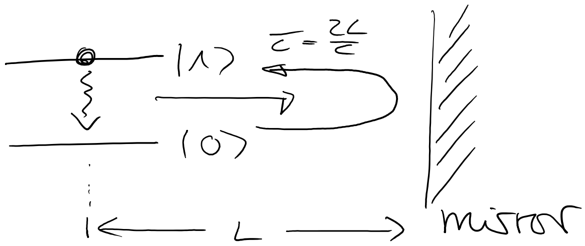
$$H = \hbar\omega_0 \sigma_{11} + \sum_k \hbar\omega_k r_k^\dagger r_k + \sum_k \hbar g_k \left[ r_k^\dagger \sigma_{01} + \sigma_{10} r_k \right]$$




$$g_k = g_0 \sin(kL) \qquad g_0 = \sqrt{\frac{\Delta k\, \gamma}{\pi}}$$

[one dimensional problem, realizable in waveguides. Experiments have been done in the Blatt (Innsbruck)]

an initial value problem i.e. it is not driven by an external laser.

$$|\psi(t)\rangle = c_1(t)\,|1, vac\rangle + \sum_k c_k(t)\,|0, \dots 1_k \dots\rangle$$

$$|c_1(t)|^2 + \int dk\, |c_k(t)|^2 = 1$$

$$i\hbar \frac{d}{dt}|\psi(t)\rangle = H|\psi(t)\rangle$$

$$i\hbar \langle 1, vac| \frac{d}{dt}|\psi(t)\rangle = i\hbar \frac{d}{dt}\langle 1, vac|\psi(t)\rangle = i\hbar\, \dot{c}_1$$

$$= \langle 1, vac|H|\psi(t)\rangle = \langle 1, vac| \Big\{ \hbar\omega_0 c_1(t)\,|1, vac\rangle + \sum_k \hbar g_k c_1(t)\,|0, \dots 1_k \dots\rangle + \sum_k \hbar\omega_k c_k(t)\,|0, \dots 1_k \dots\rangle + \sum_k \hbar g_k c_k(t)\,|1, vac\rangle \Big\}$$

$$\dot{c}_1 = -i\omega_0 c_1(t) - i \sum_k g_k c_k(t)$$

$$\dot{c}_k = -i\omega_k c_k(t) - i g_k c_1(t) \quad \Big| \quad c_k(t) = e^{-i\omega_k t}\Big[\underbrace{c_k(0)}_{=0} - i g_k \int_0^t dt'\, e^{i\omega_k t'} c_1(t')\Big]$$

$$\dot{c}_1(t) = -i\omega_0 c_1(t) - \sum_k g_k^2 \int_0^t dt'\, e^{i\omega_k(t-t')} c_1(t')$$

$$= -i\omega_0 c_1(t) - \sum_k \frac{\gamma}{\pi}\Delta k\, \sin^2 kL \int_0^t dt'\, e^{i\omega_k(t-t')} c_1(t')$$

$$= -i\omega_0 c_1(t) - \sum_k \frac{\gamma}{\pi}\Delta k \left(-\frac{1}{4}\right)\left[e^{2ikL} - 1 - 1 + e^{-i2kL}\right] \quad \Big\| \quad kL = ck\frac{L}{c} = \omega_k \frac{\tau}{2}$$

$$= -i\omega_0 c_1(t) + \frac{\gamma c}{4\pi}\int dk \int_0^t dt'\, c_1(t')\left[e^{i\omega_k(t'-t+\tau)} - 2e^{i\omega_k(t-t')} + e^{i\omega_k(t'-t-\tau)}\right]$$

$$= -i\omega_0 c_1(t) + \frac{\gamma c}{4\pi}\int_0^t dt'\left[\frac{2\pi}{c}\delta(t'-(t-\tau)) - \frac{4\pi}{c}\delta(t-t') + \frac{2\pi}{c}\delta(t'-(t+\tau))\right] c_1(t')$$

$$\dot{c}_1(t) = -i\omega_0 c_1(t) - \frac{\gamma}{2} c_1(t) + \frac{\gamma}{2} c_1(t-\tau)\Theta(t-\tau)$$

$$c_1(t) = \bar{c}_1(t)\, e^{-i\omega_0 t}$$

$$\frac{d}{dt} c_1(t) = \frac{d}{dt}\left(\bar{c}_1(t) e^{-i\omega_0}\right) \qquad \dot{\bar{c}}_1 = -\frac{\gamma}{2}\bar{c}_1(t) + \frac{\gamma}{2}\Theta(t-\tau)\, c_1(t-\tau)\, e^{i\omega_0\tau}$$

(self-consistent feedback phase)

the phase determines the sign

$$\omega_0\tau = \pi(2m) \Rightarrow e^{i\omega_0\tau} = 1$$

$$\omega_0\tau = \pi(2m+1) \Rightarrow e^{i\omega_0\tau} = -1$$

it might even convert a dissipative signal into a coherent one $\quad \omega_0\tau = \pi(2m+\frac{1}{2}) \to e^{i\omega_0\tau} = i$

solve this delay-differential equation $\quad \left\|\ \bar{c}_1(t) = f(t), \quad \frac{\gamma}{2} = \Gamma\right.$

$$s f(s) - \underbrace{f(0)}_{=1} = -\Gamma f(s) + \Gamma \underbrace{\mathcal{L}\left[\Theta(t-\tau) f(t-\tau)\right]}_{= e^{-\tau s} f(s)} e^{i\omega_0\tau}$$

$$f(s) = \frac{1}{s+\Gamma - \Gamma e^{i\omega_0\tau} e^{-\tau s}} = \frac{1}{s+\Gamma}\, \underbrace{\frac{1}{1 - \frac{\Gamma}{s+\Gamma} e^{i\omega_0\tau - \tau s}}}_{\frac{1}{1-q} = \sum_{n=0}^{\infty} q^n}$$

$$= \frac{1}{s+\Gamma}\sum_{n=0}^{\infty}\left(\frac{\Gamma e^{i\omega_0\tau} e^{-\tau s}}{s+\Gamma}\right)^n = \sum_{n=0}^{\infty}\left(\Gamma e^{i\omega_0\tau}\right)^n \frac{e^{-\tau n s}}{(s+\Gamma)^{n+1}}$$

transform back into time domain $\mathcal{L}^{-1}\{f(s) e^{-\tau s}\} = f(t-\tau)\,\Theta(t-\tau)$

and $\mathcal{L}^{-1}\left\{\frac{n!}{(s+\Gamma)^{n+1}}\right\} = t^n e^{-\Gamma t}$

$$f(s) = \sum_{n=0}^{\infty}\left(\Gamma e^{i\omega_0\tau}\right)^n \frac{1}{n!}\,\frac{n!}{(s+\Gamma)^{n+1}} e^{-\tau n s}$$

$$f(t) = \sum_{n=0}^{\infty}\frac{\left(\Gamma e^{i\omega_0\tau}\right)^n}{n!}(t-n\tau)^n e^{-\Gamma(t-n\tau)}\,\Theta(t-n\tau)$$

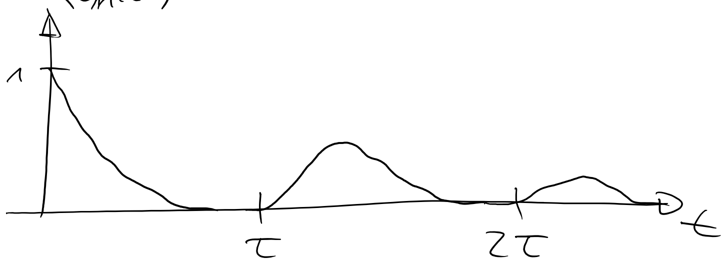
$$\bar{c}_1(t) = e^{-\Gamma t} + \Gamma e^{i\omega_0\tau}(t-\tau) e^{-\Gamma(t-\tau)}\Theta(t-\tau) + \ldots$$

excited state density $|\tilde{c}_1(t)|^2 = |e^{-i\omega_0 t} c_1(t)|^2 = |c_1(t)|^2 = \langle \sigma_{11}(t) \rangle$

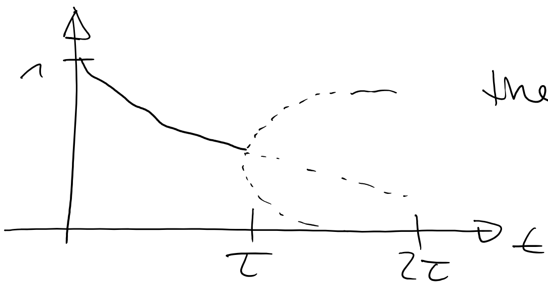
$$\langle \sigma_{11}(t) \rangle = e^{-2\Gamma t} + \Theta(t-\tau)\left\{ e^{-\Gamma t}\,\Gamma(t-\tau) e^{-\Gamma(t-\tau)} \left[ e^{-i\omega_0 \tau} + e^{i\omega_0\tau} \right] + \Gamma^2 (t-\tau)^2 e^{-2\Gamma(t-\tau)} \right\} + \dots$$

$$= e^{-\Gamma t} \left\{ 1 + \Theta(t-\tau) \left[ e^{\Gamma\tau} (\Gamma t - \Gamma\tau) \left\{ \underbrace{2\cos(\omega_0\tau)}_{\leq 2} + \underbrace{(\Gamma t - \Gamma \tau) e^{\Gamma\tau}}_{\text{large } \Gamma t \gg 1} \right\} \right] \right\}$$

$\Gamma t \gg 1$

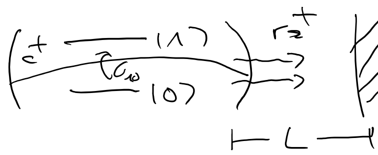



$\Gamma t \ll 1$




the phase between emission and re-excitation matters

now, we enclose the emitter with a cavity




$[c, c^\dagger] = \mathbb{1}$

$$H = \hbar\omega_0 \sigma_{11} + \hbar\omega_c c^\dagger c - \hbar M (c^\dagger \sigma_{01} + \sigma_{10} c) + \hbar \sum_z \omega_z r_z^\dagger r_z - \sum_z \hbar g_z [r_z^\dagger c + c^\dagger r_z]$$


we assume initially an excited emitter $c_1(0) = 1$

$$|\psi(t)\rangle = c_1 |1, 0, vac\rangle + c_0 |0, 1, vac\rangle + \sum_z c_z |0, 0, \dots 1_z \dots\rangle$$

$$|c_1(t)|^2 + |c_0(t)|^2 + \int dz\, |c_z(t)|^2 = 1$$

we assume also $\omega_c = \omega_0$ (resonant interaction between cavity photons and

→ go into the interaction picture                emitter)

$$i\hbar \frac{d}{dt}|\psi(t)\rangle_I = H(t)|\psi(t)\rangle_I = \Big[-M(c^+\sigma_{01} + \sigma_{10}c) - \sum_r \big(g_r(t)\, c^+ r_r + g_r^*(t)\, r_r^+ c\big)\Big]|\psi(t)\rangle_I$$

$$\dot{c}_1 = iM\, c_0(t)$$
$$\dot{c}_0 = iM\, c_1(t) + i\sum_r g_r(t)\, c_r(t)$$
$$\dot{c}_r = i\, g_r^*(t)\, c_0(t) \qquad c_r(t) - \underbrace{c_r(0)}_{=0} = \int_0^t dt'\, i\, g_r^*(t')\, c_0(t')$$

Same procedure as above

$$\dot{c}_0 = -K\, c_0(t) + K_\tau\, c_0(t-\tau)\,\Theta(t-\tau) + iM\, c_1(t), \quad K_\tau = K e^{i\omega_0 \tau}$$

solve with Laplace transform $c_1(0) = 1$, $c_0(0) = 0$

$$s\, c_1(s) - 1 = iM\, c_0(s)$$
$$s\, c_0(s) = -K\, c_0(s) + K_\tau e^{-s\tau} c_0(s) + iM\, c_1(s)$$

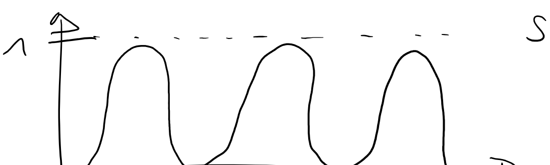
let's look first at the solution without feedback.
(this corresponds to the dissipative Jaynes-Cummings model single excitation limit)

$$c_1(s) = \frac{1}{s}\big[1 + iM\, c_0(s)\big] \qquad \| \text{ no feedback}$$

$$c_0(s) = iM \frac{1}{s^2 + Ks - M^2} = iM \frac{1}{\sqrt{M^2 - \frac{K^2}{4}}} \frac{\sqrt{M^2 - \frac{K^2}{4}}}{(s + \frac{K}{2})^2 + M^2 - \frac{K^2}{4}}$$

time domain solution $c_0(t) = iM \dfrac{\sin\left[\sqrt{M^2 - \frac{K^2}{4}}\, t\right]}{\sqrt{M^2 - \frac{K^2}{4}}}$

$$|c_0(t)|^2 = \frac{M^2}{M^2 - \frac{K^2}{4}} \sin^2\left[\sqrt{M^2 - \frac{K^2}{4}}\, t\right]$$

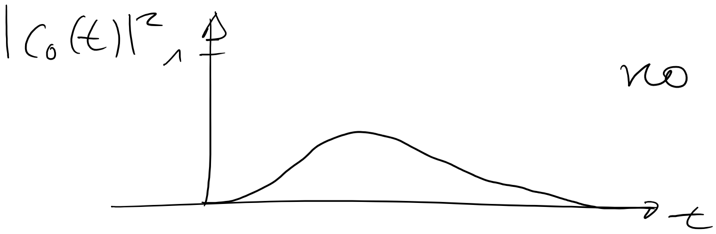



strong coupling
$M \gg K$, $M \ll \omega_0$
(but not ultrastrong due to rotating wave approximation)

medium strong coupling


what happens if $M^2 = \frac{k^2}{4}$ ?

$$c_0(s) = iM \frac{1}{(s+\frac{k}{2})^2 + \underbrace{M^2 - \frac{k^2}{4}}_{=0}} = iM \frac{1}{(s+k)^2} \longrightarrow c_0(t) = iM e^{-\frac{k}{2}t}\, t$$




no oscillations visible.

solution with feedback

$$c_0(s) = \frac{iM}{s}\left[s + \frac{k}{2} + \frac{M^2}{s} - k_\tau e^{-s\tau}\right]^{-1}$$

$$= iM \frac{1}{(s+\frac{k}{2})^2 + M^2 - \frac{k^2}{4}} \frac{1}{1 - \frac{s k_\tau e^{-s\tau}}{(s+\frac{k}{2})^2 + M^2 - \frac{k^2}{4}}}$$

$$= \frac{iM}{(s+\frac{k}{2})^2 + M^2 - \frac{k^2}{4}} \sum_{n=0}^{\infty} \left[\frac{k_\tau e^{-s\tau} s}{(s+\frac{k}{2})^2 + M^2 - \frac{k^2}{4}}\right]^n$$

let's choose again $M^2 = \frac{k^2}{4}$ !

$$c_0(s) = \frac{i\frac{k}{2}}{(s+\frac{k}{2})^2} \sum_{n=0}^{\infty} \left(\frac{k_\tau e^{-s\tau}}{s+\frac{k}{2}}\right)^n \underbrace{\left[\frac{s+\frac{k}{2}-\frac{k}{2}}{s+\frac{k}{2}}\right]^n}$$

$$= \left[1 - \frac{\frac{k}{2}}{s+\frac{k}{2}}\right]^n$$

$$= \sum_{m=0}^{n} \binom{n}{m} 1^{n-m} \frac{(-\frac{k}{2})^m}{(s+\frac{k}{2})^m}$$

$$= \frac{i\frac{k}{2}}{(s+\frac{k}{2})^2} \sum_{n=0}^{\infty} \sum_{m=0}^{n} \left[k_\tau e^{-s\tau}\right]^n \frac{n!\,(-1)^m}{m!\,(n-m)!} \frac{(\frac{k}{2})^m}{(s+\frac{k}{2})^{m+n}}$$

even in this limit, solution is very lengthy
but we know, that another non-Markovian
solution exists analytically!

let's have a look at the long time limit!

$$C_0(s) = iM\left[s^2+ks+M^2-kse^{i\omega_0\tau}e^{-s\tau}\right]^{-1}$$

we seek a pole for a s purely imaginary $s=\pm iM$

so $\left(s^2+ks+M^2-kse^{i\omega_0\tau}e^{-s\tau}\right)\Big|_{s=\pm M} \stackrel{!}{=} 0$

$$-M^2+ikM+M^2\pm ikMe^{i\omega_0\tau}e^{\mp iM\tau}=0$$

$$\boxed{e^{i\omega_0\tau \mp iM\tau} = 1} \qquad (\omega_0-M)\tau = 2\pi n$$

one option is $\tau=\frac{2\pi}{M}$ and $\omega_0\tau = 2\pi n$

$$c_0(t) = \oint C_0(s)e^{st} \stackrel{t\to\infty}{=} \sum_{poles} Res\left[C_0(s)e^{st}\right] = C_0^{lt}(t)$$

$$C_0^{lt}(t) = \sum_{\substack{m=+,-\\ s\to\pm iM}} \lim\,(s+m\,iM)\,\frac{iMe^{st}}{s^2+ks+M^2-kse^{i\omega_0\tau-s\tau}}$$

enumerator and denominator vanish.
so use L'Hôpital's rule.

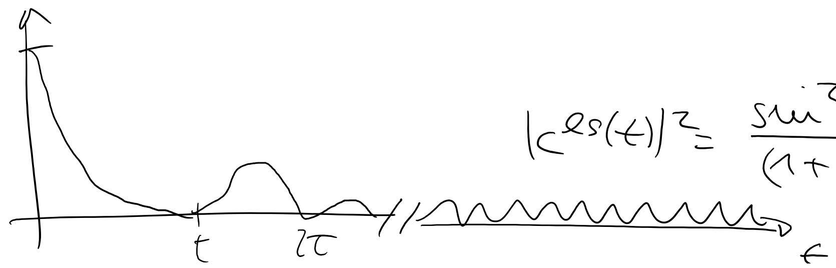
$$\frac{d}{ds}\left[(s\mp iM)e^{st}\right]\Big|_{s=\pm iM} = e^{\pm iMt}$$

$$\frac{d}{ds}\left[s^2+ks+M^2-kse^{i\omega_0\tau-s\tau}\right]\Big|_{s=iM} = 2i(M+k)$$

$$\frac{d}{ds}\left[\;\dots\;\right]\Big|_{s=-iM} = -2iM-ik2$$

$$\Rightarrow C_0^{lt}(t) = \frac{e^{iMt}\,iM}{2i(M+k)} + \frac{e^{-iMt}\,iM}{-2i(M+k)} = \frac{iM}{M+k}\sin[Mt]$$

$\Rightarrow |c_0(t)|^2$




$$|c^{ls}(t)|^2 = \frac{\sin^2(Mt)}{(1+k/M)^2}$$